

Weak Relative Pseudo-Complements of Closure Operators

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Abstract

We define the notion of weak relative pseudo-complement on meet semi-lattices, and we show that it is strictly weaker than relative pseudo-complementation, but stronger than pseudo-complementation. Our main result is that if a complete lattice \mathcal{L} is meet-continuous, then every closure operator on \mathcal{L} admits weak relative pseudo-complements with respect to continuous closure operators on \mathcal{L} .

1 Introduction

Closure operators have been widely used in various fields of theoretical computer science, like semantics of declarative languages and program analysis. In particular, Cousot and Cousot proved in [3] that the lattice of abstract interpretations, or equivalently the lattice of the semantics derivable by abstraction, for an arbitrary program whose data-objects have a lattice structure \mathcal{L} , is isomorphic to the lattice $uco(\mathcal{L})$ of the upper closure operators on \mathcal{L} . One facility offered by this result is that the meet operation in $uco(\mathcal{L})$ can be applied to compose semantics, thus getting more concrete (precise) information about the program behavior. A natural question that arises in this setting is whether it is possible to define the inverse operation, namely an operation which, starting from two semantics ρ and η for the same program, with ρ more abstract than η , gives as result the most abstract semantics φ whose composition with ρ is exactly η . Such an operation would be useful to isolate properties of ρ which are not shared by η . In terms of closure operators this amounts to define an operation analogous to the *relative pseudo-complement*, the only difference is that we require φ to be the maximum of the set of closure operators whose meet with ρ is *equal to* η , instead than *smaller than or equal to* η . It is easy to see that if the relative pseudo-complement of ρ w.r.t. η exists, then it is the intended element. The converse is not true.

We define the notion described above as a partial operation (which we call *weak relative pseudo-complement*) on arbitrary meet semi-lattices. We investigate conditions on \mathcal{L} under which pairs of elements in $uco(\mathcal{L})$ admit the weak relative pseudo-complement. In particular, we are interested in the case in which \mathcal{L} is a complete lattice (in which case also $uco(\mathcal{L})$ is a complete lattice). Our main result is that the weak relative pseudo-complement of a closure operator ρ w.r.t. η , for η continuous, always exists when \mathcal{L} is a meet-continuous complete lattice (i.e., when in \mathcal{L} the meet preserves the join of chains). In [2], this result is applied to static program analysis.

Note that the pseudo-complement is a particular case of weak relative pseudo-complement, hence the meet-continuity of \mathcal{L} also guarantees that $uco(\mathcal{L})$ is pseudo-complemented. On the other hand, the meet-continuity of \mathcal{L} is much less restrictive than the condition under which $uco(\mathcal{L})$ is relatively pseudo-complemented. In fact, in a complete lattice, relative pseudo-complementation is equivalent to complete meet-distributivity ([1]), and complete meet-distributivity in $uco(\mathcal{L})$ is equivalent to distributivity ([6]). Now, for $uco(\mathcal{L})$ to be distributive, it is necessary and sufficient that \mathcal{L} is a complete well-ordered chain ([4]). This condition is also necessary and sufficient for $uco(\mathcal{L})$ to be complemented ([7]).

2 Weak relative pseudo-complement

Before defining our notion of weak relative pseudo-complement, we first recall some basic notions and terminology. We refer to [1, 5] for all the notions recalled in the paper.

Let $\mathcal{L} = \langle L, \preceq, \wedge, \vee, \top, \perp \rangle$ be a complete lattice, where L is a set, \preceq is the partial ordering relation, \wedge is the meet, \vee is the join, \top is the top element and \perp is the bottom element. The complete lattice \mathcal{L} is *completely meet-distributive* if for each $x \in L$ and $Y \subseteq L$, $x \wedge \bigvee Y = \bigvee_{y \in Y} (x \wedge y)$ holds. Given $x, y \in L$, the *relative pseudo-complement* of x w.r.t. y , if it exists, is the (unique) element $x * y \in L$ such that $x \wedge x * y \preceq y$, and for each $z \in L$, if $x \wedge z \preceq y$ then $z \preceq x * y$. If $x * y$ exists for every $x, y \in L$, then \mathcal{L} is *relatively pseudo-complemented*. It is well known that this property holds iff \mathcal{L} is completely meet-distributive. The element $x * \perp$, if it exists, is called the *pseudo-complement* of x , and it is denoted by x^* . If x^* exists for every $x \in L$, then \mathcal{L} is *pseudo-complemented*. Obviously, (relative) pseudo-complementation is definable for a mere meet semi-lattice.

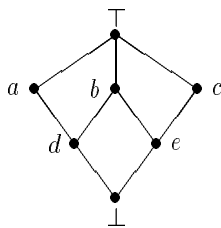
Definition 2.1 Let $\mathcal{L} = \langle L, \preceq, \wedge \rangle$ be a meet semi-lattice, and let $x, y \in L$ with $y \preceq x$. The *weak relative pseudo-complement* of x w.r.t. y , if it exists, is the (unique) element $wr(x, y) \in L$ such that

- (i) $x \wedge wr(x, y) = y$,
- (ii) for each $z \in L$, if $x \wedge z = y$ then $z \preceq wr(x, y)$.

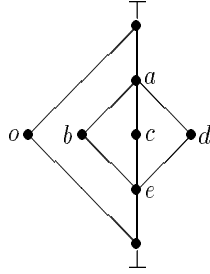
If $wr(x, y)$ exists for every $x, y \in L$ such that $y \preceq x$, then \mathcal{L} is *weakly relatively pseudo-complemented*. \square

The reason why we require $y \preceq x$, in the definition of $wr(x, y)$, is that for $y \not\preceq x$ Condition (i) would never be satisfied.

In the context of Definition 2.1, note that for $y \preceq x$ Condition (i) could equivalently be rewritten as $x \wedge wr(x, y) \preceq y$; the difference with the notion of relative pseudo-complement (within the case $y \preceq x$) is only that in Condition (ii) we require $x \wedge z = y$ instead of $x \wedge z \preceq y$. It is easy to see that for $y \preceq x$, if $x * y$ exists then $wr(x, y) = x * y$ (hence in that case, if \mathcal{L} is a complete lattice then $wr(x, y)$ can be defined as the join of all the elements z satisfying the premise of Condition (ii)). The converse is not true. For instance the following lattice is weakly relatively pseudo-complemented, but $a * d$ does not exist (while $wr(a, d) = b$.)



Concerning the relation with the pseudo-complement, we have that if $wr(x, \perp)$ exists, then it coincides with x^* . However, a pseudo-complemented lattice may not be weakly relatively pseudo-complemented. For instance, the following lattice is pseudo-complemented, but $wr(b, e)$ does not exist.



3 Weak relative pseudo-complements of closure operators

In this section, we study the conditions for the existence of weak relative pseudo-complements of closure operators. An (*upper*) *closure operator* on \mathcal{L} is a mapping $\rho : L \rightarrow L$ such that $x \preceq \rho(x)$, $\rho(\rho(x)) = \rho(x)$, and for every $x, y \in L$, if $x \preceq y$ then $\rho(x) \preceq \rho(y)$ (obviously, this notion can be given in general for posets). As usual, we say that ρ is *continuous* if for every chain $C \subseteq L$, $\rho(\bigvee C) = \bigvee_{x \in C} \rho(x)$ holds. A closure operator ρ is uniquely determined by the set of its fixpoints, which coincides with its image $\rho(L)$. A set $X \subseteq L$ is the set of fixpoints of some closure operator iff X contains \top and it is meet-closed, i.e. for any $Y \subseteq X$, with $Y \neq \emptyset$, $\bigwedge Y \in X$ holds.

Let $uco(L)$ be the set of all closure operators on L . We will denote by $uco(\mathcal{L})$ the complete lattice $\langle uco(L), \sqsubseteq, \sqcap, \sqcup, \mathfrak{U}, \iota \rangle$ where $\rho \sqsubseteq \eta$ iff for each $x \in L$, $\rho(x) \preceq \eta(x)$, $(\bigcap_{i \in I} \rho_i)(x) = \bigwedge_{i \in I} \rho_i(x)$, $(\bigcup_{i \in I} \rho_i)(x) = x$ iff for each $i \in I$, $\rho_i(x) = x$, and $\mathfrak{U}(x) = \top$ and $\iota(x) = x$. Note that $\rho \sqsubseteq \eta$ iff $\eta(L) \subseteq \rho(L)$. Note also that the set of fixpoints of $\bigcap_{i \in I} \rho_i$ is the smallest meet-closed set which contains all the fixpoints of each ρ_i , and that the set of fixpoints of $\bigcup_{i \in I} \rho_i$ is the intersection of all the sets of fixpoints of the ρ_i 's, i.e. $(\bigcup_{i \in I} \rho_i)(L) = \bigcap_{i \in I} \rho_i(L)$.

We recall that a complete lattice \mathcal{L} is *meet-continuous* if for any chain $C \subseteq L$, and for each element $x \in L$, $x \wedge \bigvee C = \bigvee_{y \in C} (x \wedge y)$ holds. It is worth noting that meet-continuity is strictly weaker than complete meet-distributivity. For instance, any finite lattice is meet-continuous, but not necessarily completely meet-distributive (i.e. distributive). Also an infinite lattice can be meet-continuous without being completely meet-distributive: consider, for instance, the ordinal sum of an ω -chain and a finite non-distributive lattice.

The following theorem expresses the main result of this paper. We prove that if \mathcal{L} is a meet-continuous complete lattice, then $uco(\mathcal{L})$ admits weak relative pseudo-complements w.r.t. continuous closure operators. In the proof, ordinal numbers will be denoted by greek letters α, β, \dots

Theorem 3.1 *If \mathcal{L} is a complete, meet-continuous lattice then for every $\rho, \eta \in uco(L)$ such that $\eta \sqsubseteq \rho$ and η is continuous, there exists $wr(\rho, \eta)$.*

Proof. For $\rho, \eta \in uco(L)$ such that $\eta \sqsubseteq \rho$ and η is continuous, let $P_{\rho\eta} = \{\varphi \in uco(L) : \rho \sqcap \varphi = \eta\}$, and define $\bar{\rho} = \sqcup P_{\rho\eta}$. Notice that $\bar{\rho}(L) = \bigcap \{\varphi(L) : \varphi \in P_{\rho\eta}\}$. We prove that $\bar{\rho} = wr(\rho, \eta)$.

Clearly $\varphi \sqsubseteq \bar{\rho}$ for every $\varphi \in P_{\rho\eta}$. Furthermore, $\eta \sqsubseteq \rho \sqcap \bar{\rho}$ holds, because $P_{\rho\eta}$ contains η . Hence, we have only to show that $\rho \sqcap \bar{\rho} \sqsubseteq \eta$, which amounts to prove that $\eta(L) \subseteq (\rho \sqcap \bar{\rho})(L)$.

Assume that there exists $\bar{x} \in \eta(L)$ such that $\bar{x} \notin (\rho \sqcap \bar{\rho})(L)$. We show that for each ordinal α we are able to construct a strictly increasing chain $\{x_\beta\}_{\beta < \alpha}$ in $\eta(L)$. Since the cardinality of this chain is $|\alpha|$, this leads to a contradiction when $|\alpha| > |\eta(L)|$.

More precisely, we prove by transfinite induction on α that it is possible to construct in $\eta(L)$ two sets $\{x_\beta\}_{\beta < \alpha}$, with $x_0 = \bar{x}$, and $\{y_\beta\}_{\beta < \alpha}$, such that:

- (i) $\{x_\beta\}_{\beta < \alpha}$ is a strictly increasing chain in $\eta(L) \setminus (\rho \sqcap \bar{\rho})(L)$,
- (ii) $\forall \beta < \alpha. y_\beta \in \rho(L)$,

- (iii) $x_0 = x_\alpha \wedge (\bigwedge_{\beta < \alpha} y_\beta)$, and
- (iv) $\forall \beta, \gamma. \beta \leq \gamma < \alpha \Rightarrow x_\beta \prec y_\gamma$.

Let us consider the three cases $\alpha = 0$, α successor ordinal, and α limit ordinal.

($\alpha = 0$) Define $x_0 = \bar{x}$. Then $x_0 \in \eta(L) \setminus (\rho \sqcap \bar{\rho})(L)$ by definition, thus (i) is satisfied. Conditions (ii), (iii) and (iv) are vacuously satisfied.

($\alpha + 1$) By the induction hypothesis (i), $x_\alpha \notin \bar{\rho}(L)$, hence there exists a closure operator $\varphi \in P_{\rho\eta}$ such that $x_\alpha \notin \varphi(L)$. Also, again by the induction hypothesis (i), $x_\alpha \notin \rho(L)$ and $x_\alpha \in \eta(L)$. By the latter and the fact that $\rho \sqcap \varphi = \eta$, we derive $x_\alpha \in (\rho \sqcap \varphi)(L)$. Therefore, there must exist $x \in \varphi(L) \setminus \rho(L)$ and $y \in \rho(L) \setminus \varphi(L)$ such that $x_\alpha = x \wedge y$. Define $x_{\alpha+1} = x$ and $y_\alpha = y$, and observe that (ii) and (iv) are satisfied.

We prove now (iii). By the induction hypothesis (iii), we have $x_0 = x_\alpha \wedge (\bigwedge_{\beta < \alpha} y_\beta)$, hence we derive $x_0 = x_{\alpha+1} \wedge y_\alpha \wedge (\bigwedge_{\beta < \alpha} y_\beta) = x_{\alpha+1} \wedge (\bigwedge_{\beta < \alpha+1} y_\beta)$.

Concerning (i), observe that $x_\alpha \preceq x_{\alpha+1}$, and $x_\alpha \neq x_{\alpha+1}$, since $x_\alpha \notin \varphi(L)$ while $x_{\alpha+1} \in \varphi(L)$. Furthermore, $x_{\alpha+1} \in \eta(L)$ since $\varphi(L) \subseteq \eta(L)$. Finally, note that the term $\bigwedge_{\beta < \alpha+1} y_\beta$ in (iii) is an element of $\rho(L)$, being $\rho(L)$ meet-closed. Since $\rho(L) \subseteq (\rho \sqcap \bar{\rho})(L)$, and $(\rho \sqcap \bar{\rho})(L)$ is meet-closed, we deduce that $x_{\alpha+1}$ cannot belong to $(\rho \sqcap \bar{\rho})(L)$, otherwise also x_0 would.

(α limit ordinal) Define $x_\alpha = \bigvee_{\beta < \alpha} x_\beta$. Conditions (ii) and (iv) are trivially satisfied by the corresponding induction hypothesis.

Consider now (iii). By the induction hypothesis (iii), we have that, for any $\beta < \alpha$, $x_0 = x_\beta \wedge (\bigwedge_{\gamma < \beta} y_\gamma)$ holds. Hence, we derive $x_0 = \bigvee_{\beta < \alpha} (x_\beta \wedge (\bigwedge_{\gamma < \beta} y_\gamma))$. By the induction hypothesis (iv), for $\beta \leq \gamma < \alpha$, $x_\beta \preceq y_\gamma$ holds. Therefore, we have $x_0 = \bigvee_{\beta < \alpha} (x_\beta \wedge (\bigwedge_{\beta \leq \gamma < \alpha} y_\gamma) \wedge (\bigwedge_{\gamma < \beta} y_\gamma)) = \bigvee_{\beta < \alpha} (x_\beta \wedge (\bigwedge_{\gamma < \alpha} y_\gamma))$, from which we get, by meet-continuity, $x_0 = (\bigvee_{\beta < \alpha} x_\beta) \wedge (\bigwedge_{\gamma < \alpha} y_\gamma) = x_\alpha \wedge (\bigwedge_{\gamma < \alpha} y_\gamma)$.

Finally, we show (i). Since all the elements of $\{x_\beta\}_{\beta < \alpha}$ are distinct, also x_α is strictly greater than x_β , for any $\beta < \alpha$. Furthermore, $x_\alpha \in \eta(L)$: in fact $\eta(x_\alpha) = \eta(\bigvee_{\beta < \alpha} x_\beta) =$ (by continuity of η) $= \bigvee_{\beta < \alpha} \eta(x_\beta) =$ (since, by the induction hypothesis (i), $x_\beta \in \eta(L)$, i.e. $\eta(x_\beta) = x_\beta$) $= \bigvee_{\beta < \alpha} x_\beta = x_\alpha$. The proof that $x_\alpha \notin (\rho \sqcap \bar{\rho})(L)$ is like in the case of α successor ordinal. ■

In the above theorem, the meet-continuity of \mathcal{L} and the continuity of η play a fundamental role. The following example shows that if η is not continuous, then $wr(\rho, \eta)$ might not exist, even if \mathcal{L} is meet-continuous.

Example 3.2 Consider the complete, meet-continuous lattice \mathcal{L} obtained as direct product of the two elements chain $\{0, 1\}$ and the $(\omega + 1)$ -chain $\mathbb{N}^\omega = \{0, 1, \dots, \omega\}$. Consider the closure operators ρ and η such that $\rho(L) = \{\langle 1, x \rangle : x \in \mathbb{N}^\omega\}$, and $\eta(L) = L \setminus \{\langle 0, \omega \rangle\}$. Clearly, η is not continuous. For any $k \in \mathbb{N}$, consider the closure operator ρ_k such that $\rho_k(L) = \{\langle 0, n \rangle : n \in \mathbb{N}, k \leq n\} \cup \{\langle 1, \omega \rangle\}$, and observe that for each k , $\rho \sqcap \rho_k = \eta$. Define $\hat{\rho} = \sqcup_{k \in \mathbb{N}} \rho_k$, and observe that $wr(\rho, \eta)$, if it exists, must be greater than or equal to $\hat{\rho}$. But $\hat{\rho}(L) = \{\top\}$, and therefore $\rho \sqcap \hat{\rho} = \rho$, which is strictly greater than η . □

Vice-versa, if \mathcal{L} is not meet-continuous, then $wr(\rho, \eta)$ might not exist, even if η is continuous.

Example 3.3 Consider the lattice \mathcal{L} defined as in the previous example, but with the element $\langle 0, \omega \rangle$ removed. In this case, \mathcal{L} is not meet-continuous. The same lattice is used in [8, page 16] as an example of a distributive complete lattice which is not pseudo-complemented. To see that $uco(\mathcal{L})$ is not weakly relatively pseudo-complemented just define ρ , the ρ_k 's, and $\hat{\rho}$ as in the previous example, and observe that for any k , we have $\rho \sqcap \rho_k = \iota$, but $\rho \sqcap \hat{\rho} = \rho$, namely $wr(\rho, \iota)$ does not exist. □

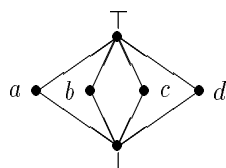
The following are easy consequences of Theorem 3.1.

Corollary 3.4 *If \mathcal{L} is meet-continuous then $uco(\mathcal{L})$ is pseudo-complemented.*

Corollary 3.5 *If \mathcal{L} does not have infinite ascending chains, then for any $\rho, \eta \in uco(L)$, with $\eta \sqsubseteq \rho$, there exists $wr(\rho, \eta)$.*

One might, at this point, raise the question whether $uco(\mathcal{L})$ satisfies the Stone identity (namely, for every $\rho \in uco(L)$, $\rho^* \sqcup \rho^{**} = \mathcal{U}$), in case the conditions for it to be pseudo-complemented hold. Next example answers this question negatively.

Example 3.6 Let \mathcal{L} be the following lattice:



Consider the closure operator ρ on \mathcal{L} for which $\rho(L) = \{\top, a, b, \perp\}$. It is clear that $\rho^*(L) = \{\top, c, d, \perp\}$ and $\rho^{**} = \rho$. Thus, the Stone identity does not hold since $(\rho^* \sqcup \rho^{**})(L) = \{\top, \perp\}$. \square

We conclude by observing that the hypothesis of meet-continuity of \mathcal{L} could be weakened. Given a complete lattice $\mathcal{L} = \langle L, \preceq, \wedge, \vee, \top, \perp \rangle$, and $\eta \in uco(L)$, the structure $\eta(\mathcal{L}) = \langle \eta(L), \preceq, \wedge \rangle$ is a complete meet-sublattice of \mathcal{L} . Furthermore, $\eta(\mathcal{L})$ is indeed a complete lattice, but, in general, not a complete sublattice of \mathcal{L} , because the join in $\eta(\mathcal{L})$ may be different from \vee . However, it is easy to show that if η is continuous, then the join of chains in $\eta(\mathcal{L})$ is the same as in \mathcal{L} . Thus, in the hypotheses of Theorem 3.1, one could replace the condition of meet-continuity of \mathcal{L} (used in the limit ordinal case) with the condition of meet-continuity of $\eta(\mathcal{L})$. The present formulation of the theorem can then be retrieved as a consequence, by observing that if \mathcal{L} is meet-continuous and $\eta \in uco(L)$ is continuous, then $\eta(\mathcal{L})$ is meet-continuous (an analogous result is proved in [5] for algebraic complete lattices).

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