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Translated tori in the characteristic varieties of complex hyperplane arrangements

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Abstract

We give examples of complex hyperplane arrangements $\mathcal{A}$ for which the top characteristic variety, $V_1(\mathcal{A})$, contains positive-dimensional irreducible components that do not pass through the origin of the algebraic torus $(\mathbb{C}^*)^{|\mathcal{A}|}$. These examples answer several questions of Libgober and Yuzvinsky. As an application, we exhibit a pair of arrangements for which the resonance varieties of the Orlik-Solomon algebra are (abstractly) isomorphic, yet whose characteristic varieties are not isomorphic. The difference comes from translated components, which are not detected by the tangent cone at the origin.

Key words: Hyperplane arrangement; Characteristic variety; Translated torus

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1 Introduction

The characteristic varieties of a space X are the jumping loci of the cohomology of X with coefficients in rank 1 local systems: $V_d(X) = \{\mathbf{t} \in (\mathbb{C}^*)^{b_1(X)} \mid \dim_{\mathbb{C}} H^1(X, \mathbb{C}_{\mathbf{t}}) \geq d\}$. If X is the complement of a normal-crossing divisor in a compact Kähler manifold with vanishing first homology, then $V_d(X)$ is a finite union of torsion-translated subtori of the character torus, see [1]. If $\mathcal{A}$ is an arrangement of hyperplanes in $\mathbb{C}^\ell$, with complement $X = X(\mathcal{A})$, then the irreducible components of $V_d(\mathcal{A}) := V_d(X)$ which contain the origin can be determined combinatorially, from the intersection lattice of $\mathcal{A}$. This follows

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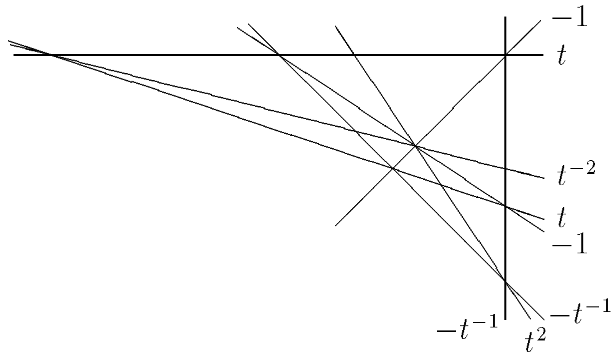


Fig. 1. Generic section of deleted B_3 arrangement and translated torus in V_1

from the fact that the tangent cone to $V_d(\mathcal{A})$ at $\mathbf{1}$ coincides with the resonance variety, $\mathcal{R}_d(\mathcal{A})$, of the Orlik-Solomon algebra (see [8] for a proof, and [18], [4], [19] for other proofs and generalizations). The variety $\mathcal{R}_d(\mathcal{A})$ in turn admits explicit combinatorial descriptions, see [11], [20].

It was first noted in [8] that there exists a hyperplane arrangement for which V_2 contains translated tori. These translated tori are isolated torsion points in V_2, lying at the intersection of several components of V_1 which do pass through the origin (see Example 4.4 in [8] and Example 3.2 below). Thus, the question arose whether the characteristic varieties of a complex hyperplane arrangement may have *positive-dimensional* translated components, see [21], Problem 5.1. In this note, we answer that question, as follows.

Theorem *There exist arrangements of complex hyperplanes for which the top characteristic variety, V_1, contains positive-dimensional irreducible components which do not pass through the origin.*

The simplest such arrangement is the "deleted B_3" arrangement, $\mathcal{D}$, discussed in Example 4.1. It is an arrangement of 8 planes in $\mathbb{C}^3$, with defining forms $x-z$, $y-z$, x, y, $x-y+z$, z, $x-y-z$, $x-y$. The arrangement $\mathcal{D}$ is fiber-type, with exponents $\{1,3,4\}$. The variety $V_1(\mathcal{D})$ has a component parametrized by $\{(t,-t^{-1},-t^{-1},t,t^2,-1,t^{-2},-1) \mid t \in \mathbb{C}^*\}$. This is a 1-dimensional torus, translated by a second root of $\mathbf{1}$. Figure 1 depicts the real part of a generic 2-dimensional section of $\mathcal{D}$ (obtained by setting $z = 2x + 3y + 1$), together with the local system corresponding to the point $t \in \mathbb{C}^*$.

The deleted B_3 arrangement can also be used to answer Conjecture 4.4 from [20], and Problems 5.2 and 5.3 from [21].

As noted in [18], [20] (see also [11]), all the components of V_1 passing through the origin must have dimension at least 2. On the other hand, all the positive-dimensional translated components that we find correspond to $\mathcal{D}$ sub-arrangements, and so have dimension 1. We do not know whether translated components can have dimension greater than 1, but we exhibit an arrangement where V_1 has 0-dimensional components (Example 5.1).

The translated components in the characteristic varieties of an arrangement $\mathcal{A}$ are not detected by the tangent cone at the origin, and thus contain information which is not available from the Orlik-Solomon algebra of $\mathcal{A}$, at least not directly. We illustrate this phenomenon in Example 5.3, where we find a pair of arrangements for which the resonance varieties are (abstractly) isomorphic, but the characteristic varieties have a different number of components. These two arrangements have non-isomorphic lattices, though. Thus, it is still an open question whether the translated components of $V_d(\mathcal{A})$ are combinatorially determined.

One of the main motivations for the study of characteristic varieties of a space X is the very precise information they give about the homology of finite abelian covers of X, see [17], [26]. From that point of view, the existence of translated components in V_1 has immediate repercussions on the Betti numbers of some finite covers of X (those corresponding to torsion characters belonging to that component). But it also affects the torsion coefficients of some abelian covers of X, and the number of certain metabelian covers of X. These aspects are pursued in joint work with D. Matei, [24]. The starting point of that paper was the discovery of 2-torsion in the homology of certain 3-fold covers of $X(\mathcal{D})$. We were led to the translated component in $V_1(\mathcal{D})$ by an effort to explain that unexpected torsion.

2 Characteristic varieties and hyperplane arrangements

We start by reviewing methods for computing the fundamental group, the characteristic varieties, and the resonance varieties of a complex hyperplane arrangement.

2.1 Characteristic varieties

Let X be a space having the homotopy type of a connected, finite CW-complex. For simplicity, we will assume throughout that $H = H_1(X, \mathbb{Z})$ is torsion free. Set $n = b_1(X)$, and fix a basis $\{t_1, \dots, t_n\}$ for $H \cong \mathbb{Z}^n$. Let $G = \pi_1(X)$ be the fundamental group, and $\mathrm{ab} : G \to H$ the abelianization homomorphism.

Let $\mathbb{C}^*$ be the multiplicative group of units in $\mathbb{C}$, and let $\mathrm{Hom}(G, \mathbb{C}^*)$ be the group of characters of G. Notice that $\mathrm{Hom}(G, \mathbb{C}^*)$ is isomorphic to the affine algebraic group $\mathrm{Hom}(H, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$, with coordinate ring $\mathbb{C}H \cong \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. For each integer $d \geq 0$, set

$$V_d(X) = \{\mathbf{t} = (t_1, \dots, t_n) \in (\mathbb{C}^*)^n \mid \dim_{\mathbb{C}} H^1(G, \mathbb{C}_{\mathbf{t}}) \geq d\},$$

where $\mathbb{C}_{\mathbf{t}}$ is the G-module $\mathbb{C}$ given by the representation $G \xrightarrow{\mathrm{ab}} \mathbb{Z}^n \xrightarrow{\mathbf{t}} \mathbb{C}^*$.

Then $V_d(X)$ is an algebraic subvariety of the complex n-torus, called the *d-th characteristic variety* of X. The characteristic varieties form a descending tower, $(\mathbb{C}^*)^n = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{n-1} \supseteq V_n$, which depends only on the isomorphism type of $G = \pi_1(X)$, up to a monomial change of basis in $(\mathbb{C}^*)^n$, see [22].

As shown in [15], the characteristic varieties of X may be interpreted as the determinantal varieties of the Alexander matrix of the group $G = \pi_1(X)$. Given a presentation $G = \langle x_1, \dots, x_m \mid r_1, \dots, r_s \rangle$, the Alexander matrix is the $s \times m$ matrix $A = \left(\frac{\partial r_i}{\partial x_j}\right)^{\mathrm{ab}}$, with entries in $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, obtained by abelianizing the Jacobian of Fox derivatives of the relations. Let $A(\mathbf{t})$ be the evaluation of A at $\mathbf{t} \in (\mathbb{C}^*)^n$. For $0 \leq d < n$, we have

$$V_d(X) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \operatorname{rank} A(\mathbf{t}) < m - d\}.$$

Remarkably, the existence of certain analytic or geometric structures on a space puts strong qualitative restrictions on the nature of its characteristic varieties. There are several results along these lines, due to Green, Lazarfeld, Simpson, and Arapura. The result we need is the following:

Theorem (Arapura [1]) *Let X be the complement of a normal-crossing divisor in a compact Kähler manifold with vanishing first homology. Then each characteristic variety $V_d(X)$ is a finite union of torsion-translated subtori of the algebraic torus $(\mathbb{C}^*)^{b_1(X)}$.*

2.2 Fundamental groups of arrangements

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of (affine) hyperplanes in $\mathbb{C}^\ell$, $\ell \geq 2$, with complement $X(\mathcal{A}) = \mathbb{C}^\ell \backslash \bigcup_{i=1}^n H_i$. We review the procedure for finding the braid monodromy presentation of the fundamental group of the complement, $G(\mathcal{A}) = \pi_1(X(\mathcal{A}))$. This presentation is equivalent to the Randell-Arvola presentation (see [6]), and the 2-complex modelled on it is homotopy-equivalent to $X = X(\mathcal{A})$ (see [16]). Since we are only interested in $G = G(\mathcal{A})$, the well-known Lefschetz-type theorem of Hamm and Lê allows us to assume that $\ell = 2$, by replacing $\mathcal{A}$ with a generic 2-dimensional slice, if necessary.

Let $v_1, \dots, v_s$ be the intersection points of the lines of $\mathcal{A}$. The combinatorics of the arrangement is encoded in its intersection poset, $L(\mathcal{A}) = \{L_1(\mathcal{A}), L_2(\mathcal{A})\}$, where $L_1 = \mathbf{n} := \{1, \dots, n\}$ and $L_2 = \{I_1, \dots, I_s\}$, with $I_k = \{i \in \mathbf{n} \mid H_i \cap v_k \neq \emptyset\}$. Choosing a generic linear projection $p : \mathbb{C}^2 \to \mathbb{C}$, and a basepoint $y_0 \in \mathbb{C}$ such that $\mathrm{Re}(y_0) > \mathrm{Re}(p(v_1)) > \cdots > \mathrm{Re}(p(v_s))$ gives orderings of the lines and vertices, which we may assume coincide with the orderings specified above. Choosing also a path in $\mathbb{C}$, starting at y_0, and passing successively through $p(v_1), \dots, p(v_s)$ gives a "braided wiring diagram," $\mathcal{W}(\mathcal{A}) = \{I_1, \beta_1, I_2, \dots, \beta_{s-1}, I_s\}$, where β_k are certain braids in the Artin braid group B_n.

Let $\{A_{i,j}\}_{1 \le i < j \le n}$ be the usual generating set for the pure braid group P_n, as specified in [2]. More generally, for $I \subset \mathbf{n}$, let $A_I \in P_n$ be the full twist on the strands indexed by I. The braid monodromy presentation of $G = \pi_1(X)$ is given by:

$$G = \langle x_1, \dots, x_n \mid \alpha_k(x_i) = x_i \text{ for } i \in I_k \setminus \{\max I_k\} \text{ and } k \in \mathbf{s} \rangle, \qquad (1)$$

where each α_k is a pure braid of the form $A_{I_k}^{\delta_k} = \delta_k^{-1} A_{I_k} \delta_k$, acting on $\mathbb{F}_n = \langle x_1, \dots, x_n \rangle$ by the Artin representation $P_n \hookrightarrow \mathrm{Aut}(F_n)$. The conjugating braids δ_k may be obtained from $\mathcal{W}$, as follows.

In the case where $\mathcal{A}$ is the complexification of a real arrangement, $\mathcal{W}$ may be realized as a (planar) wiring diagram (with all $\beta_k = 1$), in the obvious way. Each vertex set $I_k \in \mathcal{W}$ gives rise to a partition $\mathbf{n} = I_k' \cup I_k \cup I_k''$ into lower, middle, and upper wires. Set $J_k = \{i \in I_k'' \mid \min I_k < i < \max I_k\}$. Then δ_k is the subword of $A_{\mathbf{n}} = \prod_{i=2}^{n} \prod_{j=1}^{i-1} A_{j,i}$, given by $\delta_k = \prod_{i \in I_k} \prod_{j \in J_k} A_{j,i}$, see [9] (and also [6]). In the general case, the braids $\beta_1, \dots, \beta_{k-1}$ must also be taken into account, see [6] for details.

2.3 Characteristic and resonance varieties of arrangements

For an arrangement $\mathcal{A}$, with complement $X = X(\mathcal{A})$, let $V_d(\mathcal{A}) := V_d(X)$. In equations, $V_d(\mathcal{A}) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid \mathrm{rank}\, A(\mathbf{t}) < n - d\}$, where A is the Alexander matrix corresponding to the presentation (1) of $G = \pi_1(X)$. By Arapura's Theorem, $V_d(\mathcal{A})$ is a finite union of torsion-translated tori in $(\mathbb{C}^*)^n$. Denote by $\check{V}_d(\mathcal{A})$ the union of those tori that pass through $\mathbf{1}$, and by $\mathcal{V}_d(\mathcal{A})$ the tangent cone of $\check{V}_d(\mathcal{A})$ at $\mathbf{1}$. Clearly, $\mathcal{V}_d(\mathcal{A})$ is a central arrangement of subspaces in $\mathbb{C}^n$. The exponential map, $\exp : \mathrm{T}_{\mathbf{1}}((\mathbb{C}^*)^n) = \mathbb{C}^n \to (\mathbb{C}^*)^n$, $\lambda_i \mapsto e^{2\pi i \lambda_i} = t_i$, takes each subspace in $\mathcal{V}_d(\mathcal{A})$ to the corresponding subtorus in $\check{V}_d(\mathcal{A})$. In equations, $\mathcal{V}_d(\mathcal{A}) = \{\lambda \in \mathbb{C}^n \mid \mathrm{rank}\, A^{(1)}(\lambda) < n - d\}$, where $A^{(1)}$ is the linearized Alexander matrix of G, see [8] (and also [23]). The variety $\mathcal{V}_d(\mathcal{A})$ (and thus, $\check{V}_d(\mathcal{A})$, too) admits a completely combinatorial description, as follows.

The d^{th} *resonance variety* of a space X is the set $\mathcal{R}_d(X)$ of cohomology classes $\lambda \in H^1(X,\mathbb{C})$ for which there is a subspace $W \subset H^1(X,\mathbb{C})$, of dimension $d+1$, such that $\lambda \cup W = 0$ (see [23]). In other words,

$$\mathcal{R}_d(X) = \{\lambda \mid \dim H^1(H^*(X,\mathbb{C}),\lambda) \geq d\}.$$

The resonance varieties of an arrangement, $\mathcal{R}_d(\mathcal{A}) := \mathcal{R}_d(X(\mathcal{A}))$, were introduced and studied in [11]. It turns out that $\mathcal{V}_d(\mathcal{A}) = \mathcal{R}_d(\mathcal{A})$, see [8], [18] for two different proofs, and [4], [19] for recent generalizations.

As seen above, the top resonance variety is the union of a subspace arrangement: $\mathcal{R}_1(\mathcal{A}) = C_1 \cup \cdots \cup C_r$. It is also known that $\dim C_i \geq 2$, $C_i \cap C_j = \{\mathbf{0}\}$ for $i \neq j$, and $\mathcal{R}_d(\mathcal{A}) = \{\mathbf{0}\} \cup \bigcup_{\dim C_i \geq d+1} C_i$, see [20]. For each $I \in L_2(\mathcal{A})$ with $|I| \geq 3$, there is a *local* component, $C_I = \{\lambda \mid \sum_i \lambda_i = 0 \text{ and } \lambda_i = 0 \text{ for } i \notin I\}$. Note that $\dim C_I = |I| - 1$, and thus $C_I \subset \mathcal{R}_{|I|-2}(\mathcal{A})$.

The non-local components also admit a description purely in terms of $L(\mathcal{A})$, see [11], [20]. A partition $\mathsf{P} = (\mathsf{p}_1 \mid \cdots \mid \mathsf{p}_q)$ of $\mathbf{n}$ is called *neighborly* if, for all $I \in L_2(\mathcal{A})$, the following holds: $|\mathsf{p}_j \cap I| \geq |I| - 1 \Longrightarrow I \subset \mathsf{p}_j$. To a neighborly partition P, there corresponds a subspace

$$C_{\mathsf{P}} = \{\lambda \mid \sum_i \lambda_i = 0\} \cap \bigcap_I \{\lambda \mid \sum_{i \in I} \lambda_i = 0\},$$

where I ranges over all vertex sets not contained in a single block of P. Results of [20] imply that, if $\dim C_{\mathsf{P}} \geq 2$, then C_{P} is a component of $\mathcal{R}_1(\mathcal{A})$. All the components of $\mathcal{R}_1(\mathcal{A})$ arise in this fashion from neighborly partitions of sub-arrangements of $\mathcal{A}$.

This completes the combinatorial description of $\mathcal{V}_d(\mathcal{A}) = \mathcal{R}_d(\mathcal{A})$, and thus, that of $\check{V}_d(\mathcal{A})$.

2.4 Decones and linearly fibered extensions

We conclude this section with two constructions which simplify in many instances the computation of the characteristic varieties of an arrangement.

The first construction associates to a central arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ in $\mathbb{C}^\ell$, an affine arrangement, $\mathcal{A}^*$, of $n-1$ hyperplanes in $\mathbb{C}^{\ell-1}$, called a *decone* of $\mathcal{A}$. Let Q be a defining polynomial for $\mathcal{A}$. Choose coordinates $(z_1, \ldots, z_\ell)$ in

$\mathbb{C}^\ell$ so that $H_n = \ker(z_\ell)$. Then, $Q^* = Q(z_1, \ldots, z_{\ell-1}, 1)$ is a defining polynomial for $\mathcal{A}^*$, and $X(\mathcal{A}) \cong X(\mathcal{A}^*) \times \mathbb{C}^*$, see [25]. It follows that:

$$V_d(\mathcal{A}) = \{\mathbf{t} \in (\mathbb{C}^*)^n \mid (t_1, \ldots, t_{n-1}) \in V_d(\mathcal{A}^*) \text{ and } t_1 \cdots t_n = 1\},$$

and so the computation of $V_d(\mathcal{A})$ reduces to that of $V_d(\mathcal{A}^*)$, see [8].

The second construction associates to an affine arrangement, $\mathcal{A}$, in $\mathbb{C}^2$, a linearly fibered arrangement, $\widehat{\mathcal{A}}$, also in $\mathbb{C}^2$, called a *big arrangement* associated to $\mathcal{A}$. The construction depends on the choice of a linear projection $\bar{p} : \mathbb{C}^2 \to \mathbb{C}$, for which no line of $\mathcal{A}$ coincides with $\bar{p}^{-1}(\text{point})$. Let $H_1, \ldots, H_n$ be the lines of $\mathcal{A}$, let $v_1, \ldots, v_s$ be their intersection points, and let $\{w_1, \ldots, w_r\} = \bar{p}(\{v_1, \ldots, v_s\})$. Then $\widehat{\mathcal{A}} = \mathcal{A} \cup \{H_{n+1}, \ldots, H_{n+r}\}$, where $H_{n+j} = \bar{p}^{-1}(w_j)$. The restriction $\bar{p} : \widehat{X} \to \mathbb{C} \setminus \{w_1, \ldots, w_r\}$ is a (linear) fibration, with fiber $\mathbb{C} \setminus \{n \text{ points}\}$. The monodromy generators, $\bar{\alpha}_1, \ldots, \bar{\alpha}_r$ may be found using a slight modification of the algorithm from [6] (see also [3]). Deform $\bar{p}$ to a generic projection $p : \mathbb{C}^2 \to \mathbb{C}$, and let $\alpha_1, \ldots, \alpha_s$ be the corresponding braid monodromy generators. Then, $\bar{\alpha}_j = \prod_{p(v_k)=w_j} \alpha_k$. The fundamental group of $\widehat{\mathcal{A}}$ is the semidirect product $\widehat{G} = \mathbb{F}_n \rtimes_{\bar{\alpha}} \mathbb{F}_r$, with presentation

$$\widehat{G} = \langle x_1, \ldots, x_n, y_1, \ldots, y_r \mid x_i^{y_j} = \bar{\alpha}_j(x_i) \rangle. \tag{2}$$

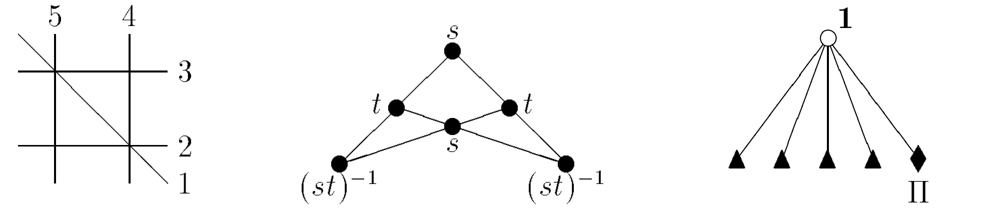
Given an arrangement $\mathcal{B}$ so that $\mathcal{B} = \widehat{\mathcal{A}}$, the presentation (2) of $\pi_1(X(\widehat{\mathcal{A}})) = \widehat{G}$ is often simpler to use than the presentation (1), obtained from the general braid monodromy algorithm applied directly to $\mathcal{B}$. In particular, if we pick $\{t_1, \ldots, t_{n+r}\} = \{x_1, \ldots, x_n, y_1, \ldots, y_r\}^{\text{ab}}$ as basis for $H_1(\widehat{G}) \cong \mathbb{Z}^{n+r}$, the Alexander matrix of $\widehat{G}$ has the block form

$$A = \begin{pmatrix} \operatorname{id} - t_{n+1} \cdot \Theta(\bar{\alpha}_1) & d_1 & \cdots & 0 \\ \vdots & & \ddots & \\ \operatorname{id} - t_{n+r} \cdot \Theta(\bar{\alpha}_r) & 0 & \cdots & d_1 \end{pmatrix}, \tag{3}$$

where $\Theta : P_n \to \mathrm{GL}(n, \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}])$ is the Gassner representation, and $d_1 = (t_1 - 1 \ \cdots \ t_n - 1)^\top$, see [7, §3.9].

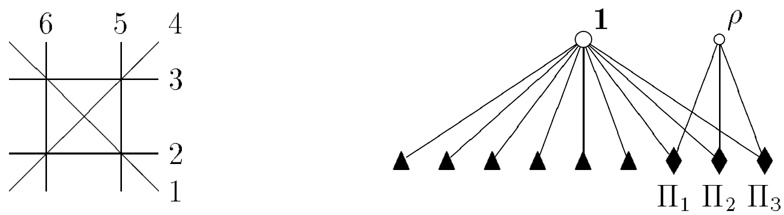
3 Warm-up Examples

We continue with some relatively simple examples of hyperplane arrangements and their characteristic varieties. These examples, which illustrate the above



(a) Decone $\mathcal{A}_3^*$ (b) Matroid and torus Π (c) Characteristic varieties

Fig. 2. The braid arrangement $\mathcal{A}_3$



(a) Decone $\mathcal{N}^*$ (b) Characteristic varieties of $\mathcal{N}$

Fig. 3. The non-Fano arrangement $\mathcal{N}$

discussion, will be useful in understanding subsequent, more complicated examples.

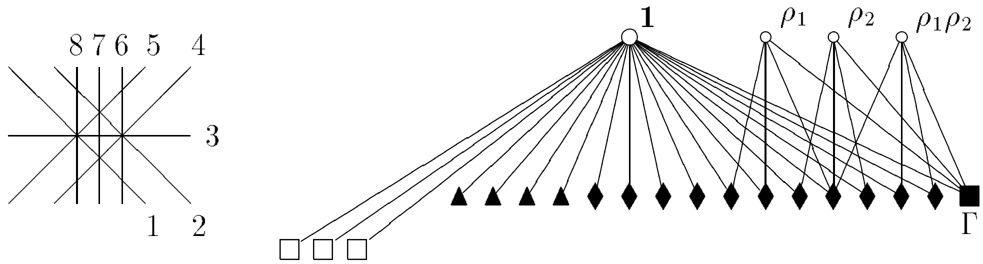
Example 3.1 Let $\mathcal{A}_3$ be the braid arrangement in $\mathbb{C}^3$, with defining polynomial $Q = xyz(x-y)(x-z)(y-z)$. The decone $\mathcal{A}_3^*$, obtained by setting $z=1$, is depicted in Figure 2(a). Note that $\mathcal{A}_3^* = \widehat{\mathcal{A}}$, where $\mathcal{A}$ consists of the lines marked $1,2,3$. Thus, $\mathcal{A}_3$ is fiber-type, with exponents $\{1,2,3\}$, and $G^* = \mathbb{F}_3 \rtimes_{\bar{\alpha}} \mathbb{F}_2$, where $\bar{\alpha}_1 = A_{12}$, $\bar{\alpha}_2 = A_{13}$ (of course, $G = P_4 \cong G^* \times \mathbb{Z}$).

The resonance and characteristic varieties of $\mathcal{A}_3$ were computed in [11], [18], [8]. The variety $V_1(\mathcal{A}_3) \subset (\mathbb{C}^*)^6$ has 4 local components, corresponding to the triple points $124, 135, 236, 456$, and one essential component, corresponding to the neighborly partition $(16|25|34)$:

$$\Pi = \{(s,t,(st)^{-1},(st)^{-1},t,s) \mid s,t \in \mathbb{C}^*\},$$

see Figure 2(b). The components of V_1 meet only at $\mathbf{1}$. Moreover, $V_2 = \cdots = V_6 = \{\mathbf{1}\}$. The intersection poset of the characteristic varieties of $\mathcal{A}_3$ is depicted in Figure 2(c). The poset is ranked by dimension (indicated by relative height), and filtered according to depth in the characteristic tower (indicated by color: V_1 in black, V_2 in white).

Example 3.2 A realization of the non-Fano plane is the arrangement $\mathcal{N}$, with defining polynomial $Q = xyz(x-y)(x-z)(y-z)(x+y-z)$. A decone $\mathcal{N}^*$ is depicted in Figure 3(a).



(a) Decone $\mathcal{B}_3^*$ (b) Characteristic varieties of $\mathcal{B}_3$

Fig. 4. The B_3 reflection arrangement

The characteristic varieties of $\mathcal{N}$ were computed in [8] (see also [21]). The variety $V_1 \subset (\mathbb{C}^*)^7$ has 6 local components, corresponding to triple points, and 3 non-local components, $\Pi_1 = \Pi(25|36|47)$, $\Pi_2 = \Pi(17|26|35)$, $\Pi_3 = \Pi(14|23|56)$, corresponding to braid sub-arrangements. The local components meet only at $\mathbf{1}$, but the non-local components also meet at the point $\rho = (1,-1,-1,1,-1,-1,1)$. The variety $V_2 = \{\mathbf{1}, \rho\}$ is a discrete algebraic subgroup of $(\mathbb{C}^*)^7$, isomorphic to $\mathbb{Z}_2$. The characteristic intersection poset of $\mathcal{N}$ is depicted in Figure 3(b).

Example 3.3 Let $\mathcal{B}_3$ be the reflection arrangement of type B_3, with defining polynomial $Q = xyz(x-y)(x-z)(y-z)(x-y-z)(x-y+z)(x+y-z)$. A decone is shown in Figure 4(a). Note that $\mathcal{B}_3^* = \widehat{\mathcal{A}}$, where $\mathcal{A}$ consists of the lines marked $1, \dots, 5$. Thus, $\mathcal{B}_3$ is fiber-type, with exponents $\{1,3,5\}$, and $G^* = \mathbb{F}_5 \rtimes_{\bar{\alpha}} \mathbb{F}_3$, where $\bar{\alpha}_1 = A_{234}$, $\bar{\alpha}_2 = A_{14}^{A_{24}A_{34}} A_{25}$, $\bar{\alpha}_3 = A_{35}^{A_{23}A_{25}}$.

A computation with Fox derivatives, using the techniques from §2, shows that the characteristic variety $V_1 \subset (\mathbb{C}^*)^9$ has 19 components:

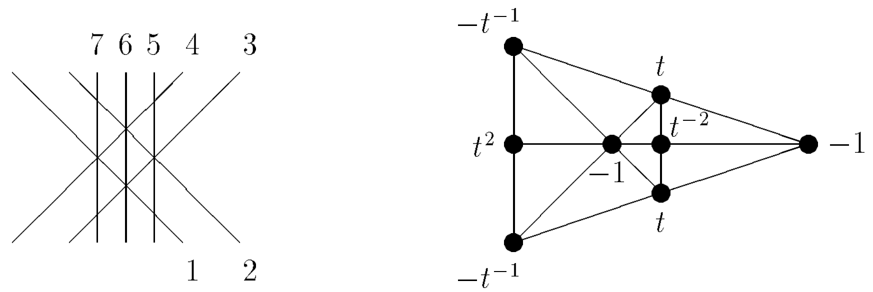
- 7 local components, corresponding to 4 triple points and 3 quadruple points.
- 11 components corresponding to braid sub-arrangements.
- 1 essential, 2-dimensional component, corresponding to the neighborly partition $(156|248|379)$, identified in [11], Example 4.6:

$$\Gamma = \{(t, s, (st)^{-2}, s, t, t^2, (st)^{-1}, s^2, (st)^{-1}) \mid s, t \in \mathbb{C}^*\}.$$

Three triples of braid components meet Γ on V_2, at the points

$$\rho_1 = (1,-1,1,-1,1,1,-1,1,-1), \quad \rho_2 = (-1,1,1,1,-1,1,-1,1,-1),$$

and $\rho_1\rho_2$. The variety V_2 consists of three 3-dimensional tori (corresponding to quadruple points), together with the discrete subgroup $\mathbb{Z}_2^2 = \{1, \rho_1, \rho_2, \rho_1\rho_2\}$. The characteristic intersection poset of $\mathcal{B}_3$ is depicted in Figure 4(b).



(a) Decone $\mathcal{D}^*$ (b) Matroid of $\mathcal{D}$ and parametrization of C

Fig. 5. The deleted B_3 arrangement $\mathcal{D}$

4 Positive-dimensional translated tori

We now come to our basic example of a complex hyperplane arrangement whose top characteristic variety contains a positive-dimensional translated component.

Example 4.1 Let $\mathcal{D}$ be the arrangement obtained from the B_3 reflection arrangement by deleting the plane $x+y-z=0$. A defining polynomial for $\mathcal{D}$ is $Q = xyz(x-y)(x-z)(y-z)(x-y-z)(x-y+z)$. The decone $\mathcal{D}^*$, obtained by setting $z=1$, is depicted in Figure 5(a). Note that $\mathcal{D}^* = \widehat{\mathcal{A}}$, where $\mathcal{A}$ consists of the lines marked $1,\dots,4$. Thus, $\mathcal{D}$ is fiber-type, with exponents $\{1,3,4\}$, and $G^* = \mathbb{F}_4 \rtimes_{\bar\alpha} \mathbb{F}_3$, where $\bar\alpha_1 = A_{23}$, $\bar\alpha_2 = A_{13}^{A_{23}} A_{24}$, $\bar\alpha_3 = A_{14}^{A_{24}}$.

The Alexander matrix of G^*, given by (3), is row-equivalent to

$$A = \begin{pmatrix}
1-t_5 & 0 & 0 & 0 & t_1-1 & 0 & 0 \\
0 & t_5(t_3-1) & 1-t_2t_5 & 0 & t_3-1 & 0 & 0 \\
0 & 1-t_5 & t_2(1-t_5) & 0 & t_2t_3-1 & 0 & 0 \\
0 & 0 & 0 & 1-t_5 & t_4-1 & 0 & 0 \\
t_6(t_3-1) & (t_3-1)(t_1t_6-1) & t_2(1-t_1t_6) & 0 & 0 & t_3-1 & 0 \\
1-t_6 & t_1(t_3-1)(t_6-1) & t_1t_2(1-t_6) & 0 & 0 & t_1t_3-1 & 0 \\
0 & t_6(t_4-1) & 0 & 1-t_2t_6 & 0 & t_4-1 & 0 \\
0 & 1-t_6 & 0 & t_2(1-t_6) & 0 & t_2t_4-1 & 0 \\
t_7(t_4-1) & (t_4-1)(t_1t_7-1) & 0 & t_2(1-t_1t_7) & 0 & 0 & t_4-1 \\
1-t_7 & t_1(1-t_7) & 0 & t_1t_2(1-t_7) & 0 & 0 & t_1t_2t_4-1 \\
0 & 1-t_7 & 0 & 0 & 0 & 0 & t_2-1 \\
0 & 0 & 1-t_7 & (t_3-1)(1-t_7) & 0 & 0 & t_4(t_3-1)
\end{pmatrix}.$$

Now recall that $V_1(\mathcal{D}) = \{\mathbf{t} \in (\mathbb{C}^*)^8 \mid (t_1,\dots,t_7) \in V_1(\mathcal{D}^*) \text{ and } t_1\cdots t_8 = 1\}$, where $V_1(\mathcal{D}^*)$ is the sub-variety of $(\mathbb{C}^*)^7$ defined by the ideal of 6×6 minors of the matrix A. Computing the primary decomposition of that ideal reveals

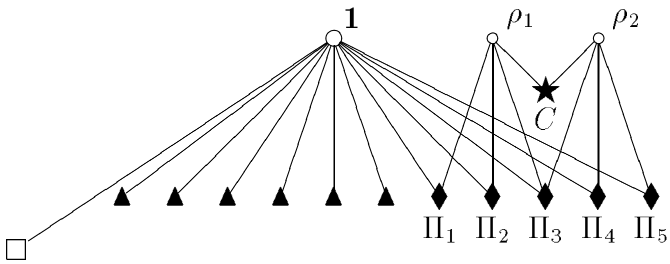


Fig. 6. The characteristic varieties of $\mathcal{D}$

that the variety $V_1(\mathcal{D})$ has 13 components:

- 7 local components, corresponding to 6 triple points and one quadruple point.
- 5 non-local components passing through **1**, corresponding to braid sub-arrangements: $\Pi_1 = \Pi(15|26|38)$, $\Pi_2 = \Pi(28|36|45)$, $\Pi_3 = \Pi(14|23|68)$, $\Pi_4 = \Pi(16|27|48)$, $\Pi_5 = \Pi(18|37|46)$.
- 1 essential component, which does not pass through **1**. This component is 1-dimensional, and is parametrized by

$$C = \{(t, -t^{-1}, -t^{-1}, t, t^2, -1, t^{-2}, -1) \mid t \in \mathbb{C}^*\}.$$

(Note that the translated torus C is one of the two connected components of $\Gamma \cap \{t_3 = 1\}$.) The braid components of $V_1(\mathcal{D})$ meet C at the points

$$\rho_1 = \Pi_1 \cap \Pi_2 \cap \Pi_3 \cap C = (1, -1, -1, 1, 1, -1, 1, -1),$$
$$\rho_2 = \Pi_3 \cap \Pi_4 \cap \Pi_5 \cap C = (-1, 1, 1, -1, 1, -1, 1, -1),$$

both of which belong to $V_2(\mathcal{D})$. The characteristic intersection poset of $\mathcal{D}$ is depicted in Figure 6.

As noted in the Introduction, this example answers Problem 5.1 in [21]. It also answers Problems 5.2 and 5.3 in [21]. Indeed, let $\lambda = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. Clearly, λ, and all its integral translates, do not belong to $\mathcal{R}_1(\mathcal{D})$, because all components of $\mathcal{R}_1(\mathcal{D})$ are non-essential. Hence, $H^1(H^*(X, \mathbb{C}), \lambda + N) = 0$, for all $N \in \mathbb{Z}^8$. On the other hand, $\mathbf{t} := \exp(\lambda) = (i, i, i, i, -1, -1, -1, -1)$ belongs to C, and thus $\dim H^1(X, \mathbb{C}_{\mathbf{t}}) = 1$.

Finally, this example also answers in the negative Conjecture 4.4 from [20], at least in its strong form. Indeed, there are infinitely many $\mathbf{t} = \exp(\lambda) \in C$ for which

$$1 = \dim H^1(X, \mathbb{C}_{\mathbf{t}}) \neq \sup_{N \in \mathbb{Z}^8} \dim H^1(H^*(X, \mathbb{C}), \lambda + N) = 0.$$

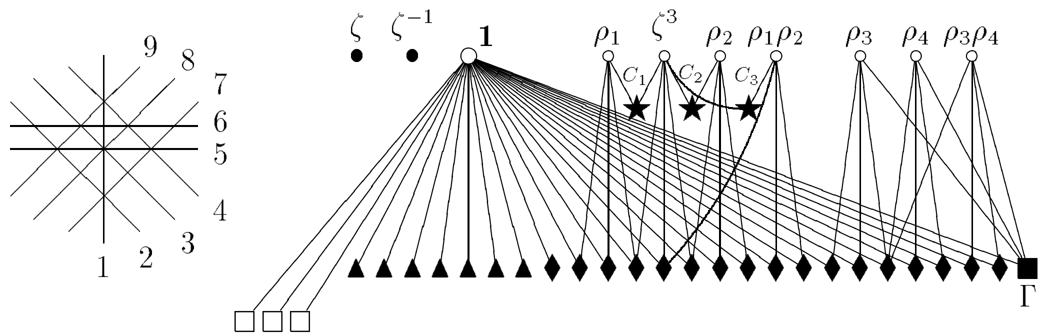


Fig. 7. The Grünbaum arrangement $\mathrm{A}_2(10)$

5 Further examples

In this section, we give a few more examples that illustrate the nature of translated components in the characteristic varieties of arrangements.

Example 5.1 Let $\mathcal{A} = \mathrm{A}_2(10)$ be the simplicial arrangement from the list in Grünbaum [14]. A defining polynomial for $\mathcal{A}$ is $Q = xyz(y-x)(y+x)(2y-z)(y-x-z)(y-x+z)(y+x+z)(y+x-z)$. Figure 7 shows a decone $\mathcal{A}^*$, together with the characteristic intersection poset of $\mathcal{A}$. The variety $V_1(\mathcal{A})$ has 33 components:

- 10 local components, corresponding to 7 triple and 3 quadruple points.
- 17 non-local components corresponding to braid sub-arrangements.
- 1 non-local component, $\Gamma = \Gamma(1510|248|379)$, corresponding to a $\mathcal{B}_3$ sub-arrangement.
- 3 components that do not pass through the origin, $C_1 = C(134568910)$, $C_2 = C(234568910)$, $C_3 = C(345678910)$, corresponding to $\mathcal{D}$ sub-arrangements.
- 2 isolated points of order 6, $\zeta = (\eta^2, \eta^2, \eta, \eta^2, \eta^2, -1, \eta^2, \eta, \eta^2, \eta)$ and ζ^{-1}, where $\eta = e^{\pi i/3}$.

Note that all positive-dimensional components of $V_1(\mathcal{A})$ are non-essential, whereas the two 0-dimensional components are essential. The non-local components meet at 7 isolated points of order 2, belonging to $V_2(\mathcal{A})$:

$$\rho_1 = (1,1,-1,-1,1,1,1,-1,-1,1), \; \rho_2 = (1,1,-1,1,-1,1,1,1,-1,-1),$$
$$\rho_3 = (-1,-1,1,-1,-1,1,1,1,1,1), \; \rho_4 = (-1,1,1,1,-1,1,-1,1,-1,1),$$

$\rho_1\rho_2$, $\rho_3\rho_4$, and ζ^3.

Example 5.2 Consider the arrangements $\mathcal{F}_1$ and $\mathcal{F}_2$, with defining polynomials $Q_1 = (x-y-z)Q$ and $Q_2 = (x-y-2z)Q$, where $Q = xyz(x-y)(y-z)(x-z)(x-2z)(x-3z)$. This pair of arrangements was introduced by Falk in [10]. Their decones and characteristic varieties are depicted in Figure 8.

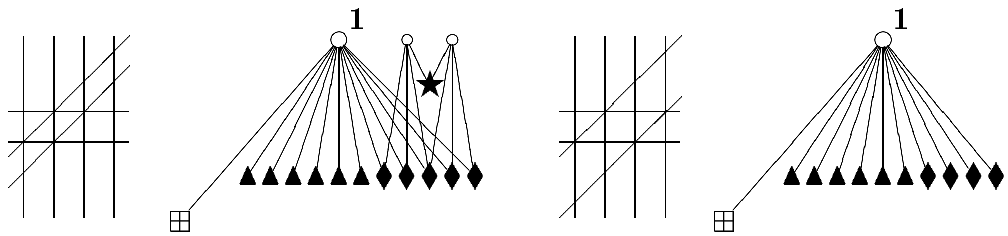


Fig. 8. The Falk fiber-type arrangements $\mathcal{F}_1$ and $\mathcal{F}_2$

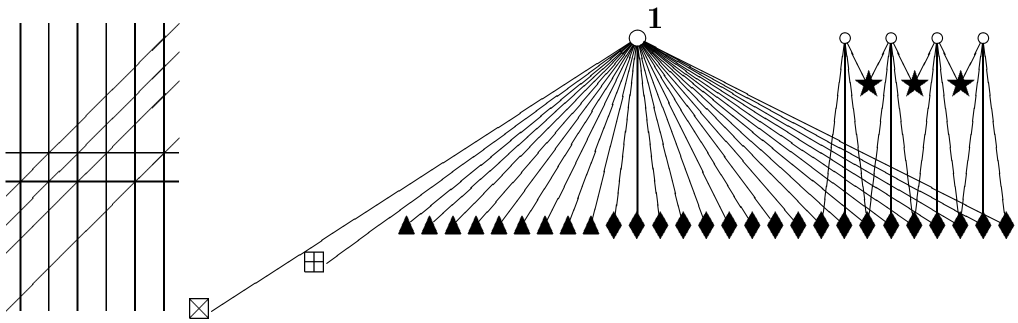


Fig. 9. The Ziegler fiber-type arrangement $\mathcal{Z}_1$

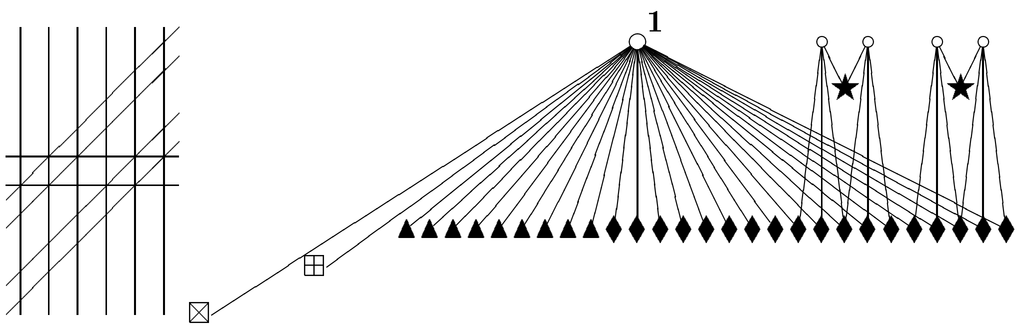


Fig. 10. The Ziegler fiber-type arrangement $\mathcal{Z}_2$

Both arrangements are fiber-type, with exponents $\{1, 4, 4\}$. Thus, by the LCS formula of Falk and Randell (see [25]), the ranks, $\phi_k(G) := \operatorname{rank} \operatorname{gr}_k(G)$, of the lower central series quotients of the two groups are the same. As noted in [5], though, the ranks, $\theta_k(G) := \operatorname{rank} \operatorname{gr}_k(G/G'')$, of the Chen groups are different: $\theta_k(G(\mathcal{F}_1)) = \frac{1}{2}(k-1)(k^2+3k+24)$ and $\theta_k(G(\mathcal{F}_2)) = \frac{1}{2}(k-1)(k^2+3k+22)$, for $k \geq 4$. Moreover, as noted in [11], the resonance varieties of the two arrangements are not isomorphic, even as abstract varieties: $\mathcal{R}_1(\mathcal{F}_1)$ has 12 components, whereas $\mathcal{R}_1(\mathcal{F}_2)$ has 11 components. An even more pronounced difference shows up in the characteristic varieties: $V_1(\mathcal{F}_1)$ has a 13^{th} component (corresponding to a sub-arrangement isomorphic to $\mathcal{D}$), which does not pass through the origin.

Example 5.3 Consider the arrangements $\mathcal{Z}_1$ and $\mathcal{Z}_2$, with defining polynomials $Q_1 = (x-y-2z)Q$ and $Q_2 = (x-y-3z)Q$, where $Q = xyz(x-y)(y-z)(x-z)(x-2z)(x-3z)(x-4z)(x-5z)(x-y-z)(x-y-4z)$. This pair of arrangements was introduced by Ziegler [27]. Their decones and characteristic varieties are depicted in Figures 9 and 10.

Both arrangements are fiber-type, with exponents $\{1, 6, 6\}$; thus, $\phi_k(G(\mathcal{Z}_1)) = \phi_k(G(\mathcal{Z}_2))$. Even more, the ranks of the Chen groups are the same: $\theta_1 = 13$, $\theta_2 = 30$, $\theta_3 = 140$, and $\theta_k = \frac{1}{24}(k-1)(k^4 + 10k^3 + 47k^2 + 86k + 696)$, for $k \geq 4$. Moreover, $\mathcal{R}_1(\mathcal{Z}_1) \cong \mathcal{R}_1(\mathcal{Z}_2)$ (as varieties), although one may show, by a rather long calculation of the respective polymatroids, that there is no linear isomorphism $\mathbb{C}^{13} \to \mathbb{C}^{13}$ taking $\mathcal{R}_1(\mathcal{Z}_1)$ to $\mathcal{R}_1(\mathcal{Z}_2)$.

On the other hand, the two groups *can* be distinguished numerically by their characteristic varieties: $V_1(\mathcal{Z}_1)$ has 32 components, whereas $V_1(\mathcal{Z}_2)$ has 31 components. Both varieties have 11 local components (corresponding to 9 triple points, 1 quintuple point, and 1 septuple point), and 18 components corresponding to braid sub-arrangements. In addition, both varieties have components which do not pass through $\mathbf{1}$, corresponding to $\mathcal{D}$ sub-arrangements: $V_1(\mathcal{Z}_1)$ has 3 such components, $V_1(\mathcal{Z}_2)$ has only 2.

6 Concluding remarks

We conclude with a few questions raised by the above examples.

Let $\mathcal{A}$ be an arrangement of n complex hyperplanes, and let $V_d(\mathcal{A}) \subset (\mathbb{C}^*)^n$ $(1 \leq d \leq n)$ be its characteristic varieties.

Question 6.1 *Are the translated components of $V_d(\mathcal{A})$ combinatorially determined?*

This problem was posed in [8] and [21], before the existence of translated components in $V_1(\mathcal{A})$ was known. Recall that $\check{V}_d(\mathcal{A})$—the union of the components of $V_d(\mathcal{A})$ passing through the identity of the torus $(\mathbb{C}^*)^n$—*is* combinatorially determined. If $\check{V}_d(\mathcal{A}) \neq V_d(\mathcal{A})$ (as in the examples from §§4–5), the question is whether $V_d(\mathcal{A}) \setminus \check{V}_d(\mathcal{A})$ is also determined by the intersection lattice of $\mathcal{A}$.

Question 6.2 *What are the possible dimensions of the translated components of $V_d(\mathcal{A})$?*

The (positive-dimensional) components passing through the origin must have dimension at least 2, and all dimensions between 2 and $n-1$ can be realized. On the other hand, at least in the examples we gave here, the components not passing through $\mathbf{1}$ have dimension either 0 or 1. The question is whether $V_d(\mathcal{A}) \setminus \check{V}_d(\mathcal{A})$ can have higher-dimensional components.

Question 6.3 *What are the possible orders of translation of the components of $V_d(\mathcal{A})$?*

In our examples, the components not passing through the origin are translated by characters of order 2 or 6. The question is whether other orders of translation can occur. Furthermore, one may ask (as a weak form of Question 6.1) whether the orders of translation are combinatorially determined. We know of a combinatorial upper bound on the lowest common multiple of these orders, but do not know when this bound is attained.

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